Name: _____

- This packet is due at **5:00 pm PDT on Wednesday, May 5**.
- Neatly fill out this packet either directly or on your own paper. Then upload your answers to Canvas. Please clearly number each problem and keep them in order.
- For each problem in this packet, **organize and show all of your work and justification** for your answer.
- Make sure that it is clear what your final answer is. This may mean boxing/circling it or it may mean doing neither.
- If you are required to draw a graph of a function, each axis must be labeled and given a scale.
- I will not be grading every individual problem, but answers will be available on Canvas after the due date.
- I am happy to help you however I can. Here are some options:
  - Come to my office hours (see syllabus).
  - Email me directly at gknapp4@uoregon.edu

1. Use the ratio test to determine whether $\sum_{n=1}^{\infty} a_n$ converges, where a_n is given in the following problems. State if the ratio test is inconclusive.

(a) $a_n = \dfrac{7^n}{n!}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\left(\frac{7^{n+1}}{(n+1)!}\right)}{\left(\frac{7^n}{n!}\right)}\right| = \lim_{n\to\infty}\frac{n!}{(n+1)!}\cdot\frac{7^{n+1}}{7^n}$$

$$= \lim_{n\to\infty}\frac{n\cdot(n-1)\cdot(n-2)\cdots 2\cdot 1}{(n+1)\cdot n\cdot(n-1)\cdots 2\cdot 1}\cdot 7$$

$$= \lim_{n\to\infty}\frac{7}{n+1} = 0 < 1$$

By the ratio test $\sum_{n=1}^{\infty}\frac{7^n}{n!}$ converges.

(b) $a_n = \dfrac{n^7}{(-3)^n}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\left(\frac{(n+1)^7}{(-3)^{n+1}}\right)}{\left(\frac{n^7}{(-3)^n}\right)}\right| = \lim_{n\to\infty}\left|\frac{(-3)^n}{(-3)^{n+1}}\cdot\frac{(n+1)^7}{n^7}\right|$$

$$= \lim_{n\to\infty}\left|\frac{1}{-3}\cdot\left(\frac{n+1}{n}\right)^7\right| = \frac{1}{3}\left(\lim_{n\to\infty}\frac{n+1}{n}\right)^7$$

$$= \frac{1}{3}\cdot 1^7 = \frac{1}{3} < 1$$

By the ratio test, $\sum_{n=1}^{\infty}\frac{n^7}{(-3)^n}$ converges.

(c) $a_n = \dfrac{(2n)!}{n^2}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\left[\frac{(2(n+1))!}{(n+1)^2}\right]}{\left[\frac{(2n)!}{n^2}\right]}\right|$$

$$= \lim_{n\to\infty}\frac{(2n+2)!}{2n!}\cdot\frac{n^2}{(n+1)^2}$$

$$= \lim_{n\to\infty}\frac{(2n+2)(2n+1)\cdot 2n\cdot(2n-1)\cdots 2\cdot 1}{(2n)\cdot(2n-1)\cdots 2\cdot 1}\cdot\frac{n^2}{(n+1)^2}$$

$$= \lim_{n\to\infty}\frac{(2n+2)(2n+1)\cdot n^2}{(n+1)^2} = \infty > 1$$ because the degree of numerator is larger than the degree of the denominator

By the ratio test, $\sum_{n=1}^{\infty}\frac{(2n)!}{n^2}$ diverges

(d) $a_n = \dfrac{2^{3n}(n!)^3}{(3n)!}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{2^{5(n+1)}\cdot((n+1)!)}{(3(n+1))!}\cdot\frac{(3n)!}{2^{5n}\cdot(n!)^3}$$

$$= \lim_{n\to\infty}\frac{2^{5n+5}}{2^{5n}}\cdot\left(\frac{(n+1)!}{n!}\right)^3\cdot\frac{(3n)!}{(3n+3)!} = \lim_{n\to\infty}2^5\cdot(n+1)^3\cdot\frac{(3n)(3n-1)\cdots 2\cdot 1}{(3n+3)(3n+2)(3n+1)\cdot 3n\cdot(3n-1)\cdots 2\cdot 1}$$

$$= \lim_{n\to\infty}\frac{32\cdot(n+1)^3}{(3n+3)(3n+2)(3n+1)} = \frac{32}{27} > 1$$

By the ratio test, $\sum_{n=1}^{\infty}\frac{2^{3n}\cdot(n!)^3}{(3n)!}$ diverges

2. Use the root test to determine whether $\sum_{n=1}^{\infty} a_n$ converges where a_n is as follows.

(a) $a_k = \left(\frac{k-1}{2k+3}\right)^k$

$\lim_{k\to\infty} |a_k|^{1/k} = \lim_{k\to\infty} \left|\frac{k-1}{2k+3}\right| = \frac{1}{2}$

By the root test, $\sum_{n=1}^{\infty} \left(\frac{n-1}{2n+3}\right)^n$ converges

(b) $a_n = \frac{(\ln n)^{2n}}{n^n}$

$\lim_{n\to\infty} \left|\frac{\ln(n)^{2n}}{n^n}\right|^{1/n} = \lim_{n\to\infty} \frac{\ln(n)^2}{n}$ $\frac{\infty}{\infty}$

$= \lim_{n\to\infty} \frac{2\ln(n)\cdot\frac{1}{n}}{1}$

$= \lim_{n\to\infty} \frac{2\ln(n)}{n}$ $\frac{\infty}{\infty}$

$= \lim_{n\to\infty} \frac{2\cdot\frac{1}{n}}{1} = \lim_{n\to\infty} \frac{2}{n} = 0$

By the root test, $\sum_{n=1}^{\infty} \frac{\ln(n)^{2n}}{n^n}$ converges

(a) ~~(x)~~ $a_n = \frac{n}{2^n}$ $\frac{(-2)^n}{n}$

$$\lim_{n\to\infty} \left|\frac{(-2)^n}{n}\right|^{1/n} = \lim_{n\to\infty} \left(\frac{2^n}{n}\right)^{1/n}$$

$$= \lim_{n\to\infty} \frac{2}{n^{1/n}} = \frac{2}{1} = 2 > 1$$

By the root test, $\sum_{n=1}^{\infty} \frac{(-2)^n}{n}$ diverges

(b) ~~(x)~~ $a_k = \frac{k^e}{e^k}$

$$\lim_{k\to\infty} \left|\frac{k^e}{e^k}\right|^{1/k} = \lim_{k\to\infty} \frac{\left(k^{1/k}\right)^e}{e} = \frac{1}{e} < 1$$

By the root test, $\sum_{k=1}^{\infty} \frac{k^e}{e^k}$ converges.

(c) ~~(x)~~ $a_k = \dfrac{1}{(1+\ln k)^k}$

$$\lim_{k\to\infty} \left| \frac{1}{(1+\ln(k))^k} \right|^{1/k} = \lim_{k\to\infty} \frac{1}{1+\ln(k)} = 0$$

By the root test, $\displaystyle\sum_{k=1}^{\infty} \frac{1}{(1+\ln(k))^k}$

converges.

(d) ~~(x)~~ $a_n = \dfrac{(\ln(1+\ln n))^n}{(\ln n)^n}$

$$\lim_{n\to\infty} \left| \frac{\ln(1+\ln(n))^n}{\ln(n)^n} \right|^{1/n} = \lim_{n\to\infty} \frac{\ln(1+\ln(n))}{\ln(n)} \quad \frac{\infty}{\infty}$$

$$= \lim_{n\to\infty} \frac{\frac{1}{1+\ln(n)} \cdot \frac{1}{n}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{1}{1+\ln(n)} = 0$$

By the root test, $\displaystyle\sum_{n=1}^{\infty} \frac{\ln(1+\ln(n))^n}{\ln(n)^n}$

converges.